

# HJB Equations for the Optimal Control of Differential Equations with Delays and State Constraints, II: Optimal Feedbacks and Approximations\*

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July 9, 2009

## Abstract

This paper, which is the natural continuation of [21], studies a class of optimal control problems with state constraints where the state equation is a differential equation with delays. This class includes some problems arising in economics, in particular the so-called models with time to build. In [21] the problem is embedded in a suitable Hilbert space  $H$  and the regularity of the associated Hamilton-Jacobi-Bellman (HJB) equation is studied. Therein the main result is that the value function  $V$  solves the HJB equation and has continuous classical derivative in the direction of the “present”. The goal of the present paper is to exploit such result to find optimal feedback strategies for the problem. While it is easy to define formally a feedback strategy in classical sense the proof of its existence and of its optimality is hard due to lack of full regularity of  $V$  and to the infinite dimension. Finally, we show some approximation results that allow us to apply our main theorem to obtain  $\varepsilon$ -optimal strategies for a wider class of problems.

**Keywords:** Hamilton-Jacobi-Bellman equation, optimal control, delay equations, verification theorem.

**A.M.S. Subject Classification:** 34K35, 49L25, 49K25.

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\*This work was partially supported by an Australian Research Council Discovery Project

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# Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>The optimal control problem</b>	<b>3</b>
2.1	Preliminary results . . . . .	5
2.2	The delay problem rephrased in infinite dimension . . . . .	6
2.2.1	Mild solutions of the state equation . . . . .	6
2.2.2	Regularity of the value function . . . . .	7
<b>3</b>	<b>Verification theorem and optimal feedback strategies</b>	<b>8</b>
3.1	The closed loop equation . . . . .	12
<b>4</b>	<b>Approximation results</b>	<b>17</b>
4.1	The case without utility on the state . . . . .	17
4.2	The case with pointwise delay in the state equation . . . . .	19
4.3	The case with pointwise delay in the state equation and without utility on the state	25

## 1 Introduction

The main purpose of this paper is to prove the existence of optimal feedback strategies for a class of optimal control problems of deterministic delay equations arising in economic models.

The paper represents the natural continuation of [21] where a class of optimal control problems with state constraints where the state equation is a differential equation with delays is studied. This class includes some problems arising in economics, in particular the so-called models with time to build. In [21] the problem is embedded in a suitable Hilbert space  $H$  and the associated Hamilton-Jacobi-Bellman (HJB) equation is studied. Therein the main result is concerned the regularity of solutions to such a HJB equation. More precisely it is shown that the value function has continuous classical derivative in the direction of the “present”. This allows to define a feedback strategy in classical sense.

In the present paper we start from this result and we exploit it to prove:

- the existence of optimal feedback strategies through a Verification Theorem;
- the existence of  $\varepsilon$ -optimal strategies for a wider family of problems through approximation results.

The class of optimal control problems is the following: given a control  $c \geq 0$  the state  $x$  satisfies the following delay equation

$$\begin{cases} x'(t) = rx(t) + f_0 \left( x(t), \int_{-T}^0 a(\xi)x(t+\xi)d\xi \right) - c(t), \\ x(0) = \eta_0, \quad x(s) = \eta_1(s), \quad s \in [-T, 0), \end{cases}$$

with state constraint  $x(\cdot) > 0$ . The objective is to maximize the functional

$$J(\eta; c(\cdot)) := \int_0^{+\infty} e^{-\rho t} [U_1(c(t)) + U_2(x(t))] dt, \quad \rho > 0,$$

over the set of the admissible controls  $c$ .

When the feedback strategy effectively exists and is admissible we prove (Theorem 3.6) that it must be optimal for the problem: this is not trivial since we do not have the full gradient of the value function and so we need to use a verification theorem for viscosity solution which is new in this context. Indeed a verification theorem in the framework of viscosity solution is given in the finite dimensional case in [31]. Adapting the technique of proof to our case is difficult due to the infinite dimensional nature of our problem and to a mistake in the key Lemma 5.2, Chapter 5 of [31] that we discovered here and that is pointed out in Remark 3.5. We then give (Proposition 3.11) sufficient conditions under which the formal optimal feedback exists and is admissible.

Since our setting (where we prove the Verification Theorem and the existence of optimal feedback strategies) do not cover the case of pointwise delay (see [21], Remark 4.8) which is used in the previously quoted applications, we go further showing three approximation results that allow to apply our main theorem to obtain  $\varepsilon$ -optimal strategies for a wider class of problems including the case of pointwise delay (Propositions 4.3, 4.6, 4.8).

The plan of the paper is as follows. Section 2 is devoted to recall the problem set in [21] and the results contained therein. Section 3 contains the first main result, i.e. the Verification Theorem 3.6. Then, in Section 3.1 we give sufficient conditions under which the hypothesis of the verification theorem is satisfied. Section 4 closes the paper with the announced approximation results.

## 2 The optimal control problem

In this section we give the setup of the optimal control problem and recall, for the reader's convenience, the main results of [21]. We will use the notations

$$L_{-T}^2 := L^2([-T, 0]; \mathbb{R}), \quad \text{and} \quad W_{-T}^{1,2} := W^{1,2}([-T, 0]; \mathbb{R}).$$

We will denote by  $H$  the Hilbert space

$$H := \mathbb{R} \times L_{-T}^2,$$

endowed with the inner product

$$\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\mathbb{R}} + \langle \cdot, \cdot \rangle_{L_{-T}^2},$$

and the norm

$$\| \cdot \|^2 = | \cdot |_{\mathbb{R}}^2 + \| \cdot \|_{L_{-T}^2}^2.$$

We will denote by  $\eta := (\eta_0, \eta_1(\cdot))$  the generic element of this space. For convenience we set also

$$H_+ := (0, +\infty) \times L_{-T}^2, \quad H_{++} := (0, +\infty) \times \{ \eta_1(\cdot) \in L_{-T}^2 \mid \eta_1(\cdot) \geq 0 \text{ a.e.} \}.$$

**Remark 2.1.** Economic motivations we are mainly interested in (see [1, 2, 27]) require to study the optimal control problem with the initial condition in  $H_{++}$ . However, the set  $H_{++}$  is not convenient to work with, since its interior with respect to the  $\| \cdot \|$ -norm is empty. That is why we enlarge the problem and allow the initial state belonging to the class  $H_+$ . ■

For  $\eta \in H_+$ , we consider an optimal control of the following differential delay equation:

$$\begin{cases} x'(t) = rx(t) + f_0\left(x(t), \int_{-T}^0 a(\xi)x(t+\xi)d\xi\right) - c(t), \\ x(0) = \eta_0, \quad x(s) = \eta_1(s), \quad s \in [-T, 0), \end{cases} \quad (1)$$

with state constraint  $x(\cdot) > 0$  and control constraint  $c(\cdot) \geq 0$ . We set up the following assumptions on the functions  $a, f_0$ .

**Hypothesis 2.2.**

- $a(\cdot) \in W_{-T}^{1,2}$  is such that  $a(\cdot) \geq 0$  and  $a(-T) = 0$ ;
- $f_0 : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$  is jointly concave, nondecreasing with respect to the second variable, Lipschitz continuous with Lipschitz constant  $C_{f_0}$ , and

$$f_0(0, y) > 0, \quad \forall y > 0. \quad (2)$$

■

**Remark 2.3.** In the papers [1, 2, 27] the pointwise delay is used. We cannot treat exactly this case for technical reason that are explained in Remark 4.7 below. However we have the freedom of choosing the function  $a$  in a wide class and this allows to take account of various economic phenomena. Moreover we can approximate the pointwise delay with suitable sequence of functions  $\{a_n\}$  getting convergence of the value function and constructing  $\varepsilon$ -optimal strategies (see Subsections 4.2 and 4.3). ■

We say that a function  $x : [-T, \infty) \rightarrow \mathbb{R}^+$  is a solution to equation (1) if  $x(t) = \eta_1(t)$  for  $t \in [-T, 0)$  and

$$x(t) = \eta_0 + \int_0^t rx(s)ds + \int_0^t f_0\left(x(s), \int_{-T}^0 a(\xi)x(s+\xi)d\xi\right)ds - \int_0^t c(s)ds, \quad t \geq 0.$$

**Theorem 2.4.** *For any given  $\eta \in H_+$ ,  $c(\cdot) \in L_{loc}^1([0, +\infty); \mathbb{R}^+)$ , equation (1) admits a unique solution that is absolutely continuous on  $[0, +\infty)$ .*

**Proof.** See [21]. □

We denote by  $x(\cdot; \eta, c(\cdot))$  the unique solution of (1) with initial point  $\eta \in H_+$  and under the control  $c(\cdot)$ . We emphasize that this solution actually satisfies pointwise only the integral equation associated with (1); it satisfies (1) in differential form only for almost every  $t \in [0, +\infty)$ .

For  $\eta \in H_+$  we define the class of the admissible controls starting from  $\eta$  as

$$\mathcal{C}(\eta) := \{c(\cdot) \in L_{loc}^1([0, +\infty); \mathbb{R}^+) \mid x(\cdot; \eta, c(\cdot)) > 0\}.$$

Setting  $x(\cdot) := x(\cdot; \eta, c(\cdot))$ , the problem consists in maximizing the functional

$$J(\eta; c(\cdot)) := \int_0^{+\infty} e^{-\rho t} [U_1(c(t)) + U_2(x(t))] dt, \quad \rho > 0,$$

over the set of the admissible strategies.

The following will be standing assumptions on the utility functions  $U_1, U_2$ , holding throughout the whole paper.

**Hypothesis 2.5.**

- (i)  $U_1 \in C([0, +\infty); \mathbb{R}) \cap C^2((0, +\infty); \mathbb{R})$ ,  $U_1' > 0$ ,  $U_1'(0^+) = +\infty$ ,  $U_1'' < 0$  and  $U_1$  is bounded.
- (ii)  $U_2 \in C((0, +\infty); \mathbb{R})$  is increasing, concave, bounded from above. Moreover

$$\int_0^{+\infty} e^{-\rho t} U_2(e^{-C_{f_0} t}) dt > -\infty. \quad (3)$$

■

Since  $U_1, U_2$  are bounded from above, the previous functional is well-defined for any  $\eta \in H_+$  and  $c(\cdot) \in \mathcal{C}(\eta)$ . We set

$$\bar{U}_1 := \lim_{s \rightarrow +\infty} U_1(s), \quad \bar{U}_2 := \lim_{s \rightarrow +\infty} U_2(s).$$

We refer to [21] for comments on the assumptions above.

For  $\eta \in H_+$  the value function of our problem is defined by

$$V(\eta) := \sup_{c(\cdot) \in \mathcal{C}(\eta)} J(\eta, c(\cdot)), \quad (4)$$

with the convention  $\sup \emptyset = -\infty$ . The domain of the value function is the set

$$\mathcal{D}(V) := \{\eta \in H_+ \mid V(\eta) > -\infty\}.$$

Due to the assumptions on  $U_1, U_2$  we directly get that  $V \leq \frac{1}{\rho}(\bar{U}_1 + \bar{U}_2)$ .

## 2.1 Preliminary results

The proof of the following qualitative results on the value function can be found in [21].

**Lemma 2.6** (Comparison). *Let  $\eta \in H_+$  and let  $c(\cdot) \in L_{loc}^1([0, +\infty); \mathbb{R}^+)$ . Let  $x(t)$ ,  $t \geq 0$ , be an absolutely continuous function satisfying almost everywhere the differential inequality*

$$\begin{cases} x'(t) \leq rx(t) + f_0\left(x(t), \int_{-T}^0 a(\xi)x(t+\xi)d\xi\right) - c(t), \\ x(0) \leq \eta_0, \quad x(s) \leq \eta_1(s), \text{ for a.e. } s \in [-T, 0]. \end{cases}$$

*Then  $x(\cdot) \leq x(\cdot; \eta, c(\cdot))$ .*

□

**Proposition 2.7.** *We have*

$$H_{++} \subset \mathcal{D}(V), \quad \mathcal{D}(V) = \{\eta \in H_+ \mid 0 \in \mathcal{C}(\eta)\}.$$

*The set  $\mathcal{D}(V)$  is convex and the value function  $V$  is concave on  $\mathcal{D}(V)$ .*

□

**Proposition 2.8.** *We have the following statements:*

1.  $V(\eta) < \frac{1}{\rho}(\bar{U}_1 + \bar{U}_2)$  for any  $\eta \in H_+$ .
2.  $\lim_{\eta_0 \rightarrow +\infty} V(\eta_0, \eta_1(\cdot)) = \frac{1}{\rho}(\bar{U}_1 + \bar{U}_2)$ , for all  $\eta_1(\cdot) \in L_{-T}^2$ .
3.  $V$  is strictly increasing with respect to the first variable.

□

## 2.2 The delay problem rephrased in infinite dimension

Our aim is to apply the dynamic programming technique in order to solve the control problem described in the previous section. However, this approach requires a markovian setting. That is why we will reformulate the problem as an infinite-dimensional control problem. Let  $\hat{n} = (1, 0) \in H_+$  and let us consider, for  $\eta \in H$  and  $c(\cdot) \in L^1([0, +\infty); \mathbb{R}^+)$ , the following evolution equation in the space  $H$ :

$$\begin{cases} X'(t) = AX(t) + F(X(t)) - c(t)\hat{n}, \\ X(0) = \eta \in H_+. \end{cases} \quad (5)$$

In the equation above:

- $A : \mathcal{D}(A) \subset H \longrightarrow H$  is an unbounded operator defined by  $A(\eta_0, \eta_1(\cdot)) := (r\eta_0, \eta_1'(\cdot))$  on

$$\mathcal{D}(A) := \{\eta \in H \mid \eta_1(\cdot) \in W_{-T}^{1,2}, \eta_1(0) = \eta_0\};$$

- $F : H \longrightarrow H$  is a Lipschitz continuous map defined by

$$F(\eta_0, \eta_1(\cdot)) := (f(\eta_0, \eta_1(\cdot)), 0),$$

$$\text{where } f(\eta_0, \eta_1(\cdot)) := f_0\left(\eta_0, \int_{-T}^0 a(\xi)\eta_1(\xi)d\xi\right).$$

It is well known that  $A$  is the infinitesimal generator of a strongly continuous semigroup  $(S(t))_{t \geq 0}$  on  $H$ ; its explicit expression is given by

$$S(t)(\eta_0, \eta_1(\cdot)) = \left(\eta_0 e^{rt}, I_{[-T, 0]}(t + \cdot) \eta_1(t + \cdot) + I_{[0, +\infty)}(t + \cdot) \eta_0 e^{r(t + \cdot)}\right);$$

### 2.2.1 Mild solutions of the state equation

Here we give a definition of the *mild solution* to (5), state the existence and uniqueness of such a solution and the equivalence between the one dimensional delay problem and the infinite dimensional one. We refer to [21] for the proofs.

**Definition 2.9.** A mild solution of (5) is a function  $X \in C([0, +\infty); H)$  which satisfies the integral equation

$$X(t) = S(t)\eta + \int_0^t S(t - \tau)F(X(\tau))d\tau + \int_0^t c(\tau)S(t - \tau)\hat{n}d\tau. \quad (6)$$

**Theorem 2.10.** For any  $\eta \in H$ , there exists a unique mild solution of (5).  $\square$

We denote by  $X(\cdot; \eta, c(\cdot)) = (X_0(\cdot; \eta, c(\cdot)), X_1(\cdot; \eta, c(\cdot)))$  the unique solution to (5) for the initial state  $\eta \in H$  and under the control  $c(\cdot) \in L^1([0, +\infty); \mathbb{R}^+)$ . The following equivalence result justifies our approach.

**Proposition 2.11.** Let  $\eta \in H_+$ ,  $c(\cdot) \in \mathcal{C}(\eta)$  and let  $x(\cdot)$ ,  $X(\cdot)$  be respectively the unique solution to (1) and the unique mild solution to (5) starting from  $\eta$  and under the control  $c(\cdot)$ . Then, for any  $t \geq 0$ , we have the equality in  $H$

$$X(t) = (x(t), x(t + \xi)_{\xi \in [-T, 0]}).$$

$\square$

### 2.2.2 Regularity of the value function

Here we state the regularity properties of the value function. We refer to [21] for the proofs.

We recall that the generator  $A$  of the semigroup  $(S(t))_{t \geq 0}$  has bounded inverse in  $H$  given by

$$A^{-1}(\eta_0, \eta_1)(s) = \left( \frac{\eta_0}{r}, \frac{\eta_0}{r} - \int_s^0 \eta_1(\xi) d\xi \right), \quad s \in [-T, 0].$$

It is well known that  $A^{-1}$  is compact in  $H$ . It is also clear that  $A^{-1}$  is an isomorphism of  $H$  onto  $\mathcal{D}(A)$  endowed with the graph norm.

We define the  $\|\cdot\|_{-1}$ -norm on  $H$  by

$$\|\eta\|_{-1} := \|A^{-1}\eta\|.$$

**Proposition 2.12.** *The set  $\mathcal{D}(V)$  is open in the space  $(H, \|\cdot\|_{-1})$  and the value function is continuous with respect to  $\|\cdot\|_{-1}$  on  $\mathcal{D}(V)$ . Moreover*

$$(\eta_n) \subset \mathcal{D}(V), \quad \eta_n \rightharpoonup \eta \in \mathcal{D}(V) \implies V(\eta_n) \rightarrow V(\eta). \quad (7)$$

□

Therefore, we can apply the following result to the value function.

**Proposition 2.13.** *Let  $v : \mathcal{D}(V) \rightarrow \mathbb{R}$  be a concave function continuous with respect to  $\|\cdot\|_{-1}$ . Then*

1.  $v = u \circ A^{-1}$ , where  $u : \mathcal{O} \subset H \rightarrow \mathbb{R}$  is a concave  $\|\cdot\|$ -continuous function.
2.  $D^+v(\eta) \subset \mathcal{D}(A^*)$ , for any  $\eta \in \mathcal{D}(V)$ .
3.  $D^+u(A^{-1}\eta) = A^*D^+v(\eta)$ , for any  $\eta \in \mathcal{D}(V)$ . In particular, since  $A^*$  is injective,  $v$  is differentiable at  $\eta$  if and only if  $u$  is differentiable at  $A^{-1}\eta$ .
4. If  $\zeta \in D^*v(\eta)$ , then there exists a sequence  $\eta_n \rightarrow \eta$  such that there exist  $\nabla v(\eta_n)$ ,  $\nabla v(\eta_n) \rightarrow \zeta$  and  $A^*\nabla v(\eta_n) \rightharpoonup A^*\zeta$ .

□

The HJB equation associated to our optimization problem is

$$\rho v(\eta) = \langle \eta, A^*\nabla v(\eta) \rangle + f(\eta)v_{\eta_0}(\eta) + U_2(\eta_0) + \mathcal{H}(v_{\eta_0}(\eta)), \quad (8)$$

where  $\mathcal{H}$  is the Legendre transform of  $U_1$ , i.e.

$$\mathcal{H}(\zeta_0) := \sup_{c \geq 0} (U_1(c) - \zeta_0 c), \quad \zeta_0 > 0.$$

Due to Hypothesis 2.5-(i) and to Corollary 26.4.1 of [30], we have that  $\mathcal{H}$  is strictly convex on  $(0, +\infty)$ . Notice that, thanks to Proposition 2.8-(3),

$$D_{\eta_0}^+ V(\eta) := \{\zeta_0 \in \mathbb{R} \mid (\zeta_0, \zeta_1(\cdot)) \in D^+ V(\eta)\} \subset (0, \infty)$$

for any  $\eta \in \mathcal{D}(V)$ , i.e. where  $\mathcal{H}$  is defined.

We can study this equation following the viscosity approach. In order to do that, we have to define a suitable set of regular test functions. This is the set

$$\tau := \left\{ \varphi \in C^1(H) \mid \nabla \varphi(\cdot) \in \mathcal{D}(A^*), \eta_m \rightarrow \eta \Rightarrow A^* \nabla \varphi(\eta_m) \rightarrow A^* \nabla \varphi(\eta) \right\}. \quad (9)$$

Let us define, for  $c \geq 0$ , the operator  $\mathcal{L}^c$  on  $\tau$  by

$$[\mathcal{L}^c \varphi](\eta) := -\rho \varphi(\eta) + \langle \eta, A^* \nabla \varphi(\eta) \rangle + f(\eta) \varphi_{\eta_0}(\eta) - c \varphi_{\eta_0}(\eta).$$

**Definition 2.14.** (i) A continuous function  $v : \mathcal{D}(V) \rightarrow \mathbb{R}$  is called a viscosity subsolution of (8) on  $\mathcal{D}(V)$  if for any  $\varphi \in \tau$  and any  $\eta_M \in \mathcal{D}(V)$  such that  $v - \varphi$  has a  $\|\cdot\|$ -local maximum at  $\eta_M$  we have

$$\rho v(\eta_M) \leq \langle \eta_M, A^* \nabla \varphi(\eta_M) \rangle + f(\eta_M) \varphi_{\eta_0}(\eta_M) + U_2(\eta_0) + \mathcal{H}(\varphi_{\eta_0}(\eta_M)).$$

(ii) A continuous function  $v : \mathcal{D}(V) \rightarrow \mathbb{R}$  is called a viscosity supersolution of (8) on  $\mathcal{D}(V)$  if for any  $\varphi \in \tau$  and any  $\eta_m \in \mathcal{D}(V)$  such that  $v - \varphi$  has a  $\|\cdot\|$ -local minimum at  $\eta_m$  we have

$$\rho v(\eta_m) \geq \langle \eta_m, A^* \nabla \varphi(\eta_m) \rangle + f(\eta_m) \varphi_{\eta_0}(\eta_m) + U_2(\eta_0) + \mathcal{H}(\varphi_{\eta_0}(\eta_m)).$$

(iii) A continuous function  $v : \mathcal{D}(V) \rightarrow \mathbb{R}$  is called a viscosity supersolution of (8) on  $\mathcal{D}(V)$  if it is both a viscosity sub and supersolution.

**Theorem 2.15.** *The value function  $V$  is a viscosity solution of (8) on  $\mathcal{D}(V)$ .*  $\square$

Actually the concave  $\|\cdot\|_{-1}$ -continuous viscosity solutions of (8) (so that in particular the value function  $V$ ) are differentiable along the direction  $\hat{n} = (1, 0)$ . This is stated in the next result: we refer to [21] for the proof.

**Theorem 2.16.** *Let  $v$  be a concave  $\|\cdot\|_{-1}$ -continuous viscosity solution of (8) on  $\mathcal{D}(V)$ . Then  $v$  is differentiable along the direction  $\hat{n} = (1, 0)$  at any point  $\eta \in \mathcal{D}(V)$  and the function  $\eta \mapsto v_{\eta_0}(\eta)$  is continuous on  $\mathcal{D}(V)$ .*  $\square$

### 3 Verification theorem and optimal feedback strategies

Here we prove a Verification Theorem yielding optimal strategies for the problem. We start with the following definition.

**Definition 3.1.** Let  $\eta \in \mathcal{D}(V)$ . An admissible control  $c^*(\cdot) \in \mathcal{C}(\eta)$  is said to be optimal for the initial state  $\eta$  if  $J(\eta; c^*(\cdot)) = V(\eta)$ . In this case the corresponding state trajectory  $x^*(\cdot) := x(\cdot; \eta, c^*(\cdot))$  is said to be an optimal trajectory and the couple  $(x^*(\cdot), c^*(\cdot))$  is said an optimal couple.  $\square$

Thanks to the regularity result of the previous section we can define, at least formally, the “candidate” optimal feedback map on  $\mathcal{D}(V)$ , which is given by

$$C(\eta) := \operatorname{argmax}_{c \geq 0} (U_1(c) - c V_{\eta_0}(\eta)), \quad \eta \in \mathcal{D}(V). \quad (10)$$

Note that this map is well-defined since  $V$  is concave and, by Proposition 2.8, strictly increasing, so that we have  $V_{\eta_0}(\eta) \in (0, +\infty)$  for all  $\eta \in \mathcal{D}(V)$ . Existence and uniqueness of the  $\operatorname{argmax}$  follow from the assumptions on  $U_1$ . Moreover, since  $V_{\eta_0}$  is continuous on  $\mathcal{D}(V)$ , also  $C$  is



continuous on  $\mathcal{D}(V)$ . The closed-loop delay state equation associated with this map is, for  $\eta \in \mathcal{D}(V)$ ,

$$\begin{cases} x'(t) = rx(t) + f_0 \left( x(t), \int_{-T}^0 a(\xi)x(t+\xi)d\xi \right) - C \left( (x(t), x(t+\xi)|_{\xi \in [-T,0]}) \right), \\ x(0) = \eta_0, \quad x(s) = \eta_1(s), \quad s \in [-T, 0]. \end{cases} \quad (11)$$

Now we want to prove a Verification Theorem: if the closed loop equation (11) has a strictly positive solution  $x^*(\cdot)$ , (so that we must have  $(x^*(t), x^*(t+\xi)|_{\xi \in [-T,0]}) \in \mathcal{D}(V)$  and the term  $C \left( (x^*(t), x^*(t+\xi)|_{\xi \in [-T,0]}) \right)$  is well-defined for every  $t \geq 0$ ), then the feedback strategy

$$c^*(t) := C \left( (x^*(t), x^*(t+\xi)|_{\xi \in [-T,0]}) \right) \quad (12)$$

is optimal. Notice that, by definition of  $c^*(\cdot)$ , if  $x^*(\cdot)$  is a strictly positive solution of (11), then  $c^*(\cdot)$  is admissible and, setting  $X^*(t) := X(t; \eta, c^*(\cdot))$ , we have

$$X^*(t) = (x^*(t), x^*(t+\xi)|_{\xi \in [-T,0]}) \in \mathcal{D}(V), \quad \forall t \geq 0.$$

In order to prove a Verification Theorem, formally we need to integrate the function

$$t \mapsto \frac{d}{dt} [e^{-\rho t} V(X^*(t))]. \quad (13)$$

Thus we need something like a Fundamental Theorem of Calculus relating the function and the integral of its "derivative". Since we do not require the initial datum  $\eta$  belonging to  $\mathcal{D}(A)$  and the operator  $A$  works as a shift operator on the infinite-dimensional component, we do not have the condition  $X^*(t) \in \mathcal{D}(A)$  for almost every  $t \geq 0$  giving a regularity for the function

$$t \mapsto e^{-\rho t} V(X^*(t))$$

sufficient to apply the Fundamental Theorem of Calculus (see [28], Theorems 5.4, 5.5, Chapter 6). Therefore we can suppose only that the function (13) is continuous and we should try to apply a generalized Fundamental Theorem of Calculus in inequality form. There is such a result in [31], Lemma 5.2, Chapter 5. Unfortunately such result is not true as it is stated (see Remark 3.5 for a counterexample), so we have to refer to other results based on the theory first formulated by Dini and Lebesgue. We refer to [10, 25, 29] sketching the ideas we need. If  $g$  is a continuous function on some interval  $[\alpha, \beta] \subset \mathbb{R}$ , the right Dini derivatives of  $g$  are defined by

$$D^+g(t) = \limsup_{h \downarrow 0} \frac{g(t+h) - g(t)}{h}, \quad D_+g(t) = \liminf_{h \downarrow 0} \frac{g(t+h) - g(t)}{h}, \quad t \in [\alpha, \beta),$$

and the left Dini derivatives by

$$D^-g(t) = \limsup_{h \uparrow 0} \frac{g(t+h) - g(t)}{h}, \quad D_-g(t) = \liminf_{h \uparrow 0} \frac{g(t+h) - g(t)}{h}, \quad t \in (\alpha, \beta].$$

The key result is the following (see [10], Theorem 1.2, Chapter 4).

**Proposition 3.2.** *If  $g$  is a continuous real function on  $[\alpha, \beta]$ , then the bounds of each Dini's derivative are equal to the bounds of the set of the difference quotients*

$$\left\{ \frac{g(t) - g(s)}{t - s} \mid t, s \in [\alpha, \beta] \right\}.$$

□

An immediate consequence of Proposition 3.2 above is the following.

**Proposition 3.3** (Monotonicity result). *Let  $g \in C([\alpha, \beta]; \mathbb{R})$  be such that  $D^+g(t) \geq 0$  for all  $t \in [\alpha, \beta]$ . Then  $g$  is nondecreasing on  $[\alpha, \beta]$ .*  $\square$

Now we can give a simple lemma useful for proving the Verification Theorem.

**Lemma 3.4.** *Let  $g, \mu \in C([0, +\infty); \mathbb{R})$  such that*

$$D_-g(t) \geq \mu(t), \quad \forall t \in (0, +\infty). \quad (14)$$

*Then, for every  $0 \leq \alpha \leq \beta < +\infty$ ,*

$$g(\beta) - g(\alpha) \geq \int_{\alpha}^{\beta} \mu(t) dt. \quad (15)$$

$\square$

**Proof.** Since  $D_-g(t) \geq \mu(t)$  for every  $t \in (0, +\infty)$ , we have  $D_-[g(t) - \int_0^t \mu(s) ds] \geq 0$  for every  $t \in (0, +\infty)$ . Thanks to Proposition 3.2 we have also  $D^+[g(t) - \int_0^t \mu'(s) ds] \geq 0$  for every  $t \in [0, +\infty)$ . Therefore, due to Proposition 3.3,  $t \mapsto g(t) - \int_0^t \mu(s) ds$  is nondecreasing, getting the claim.  $\square$

**Remark 3.5.** Following [25], we give some remarks on Lemma 3.4.

- The assumption that  $\mu$  is continuous can be replaced assuming that  $\mu$  is a finite-valued (Lebesgue) measurable and integrable function (Theorem 9 of [25]); also condition (14) can be weakened assuming that it holds out of a countable set (Section 5.b of [25]).
- Condition (14) can be weakened assuming that it holds almost everywhere adding the assumption  $D_-g > -\infty$  everywhere (Section 5.c of [25]).
- We cannot further weaken (14): if it is verified only almost everywhere without any further assumption on  $D_-g$ , then (15) is no longer true. For example, if  $g = -f$  on  $[0, 1]$ , where  $f$  is the Cantor function and  $\mu \equiv 0$ , we have

$$\mu(t) = 0 = g'(t) = D_-g(t) \quad \text{for a.e. } t \in (0, 1].$$

Therefore, taking  $\alpha = 0$ ,  $\beta = 1$ , the left handside of (15) is  $-1$ , while the right handside is 0. Indeed in this case  $D_-g = -\infty$  on the Cantor set. So Lemma 5.2, Chapter 5, of [31] is not correct. Indeed the condition required therein is not sufficient to apply Fatou's Lemma in the proof.  $\blacksquare$

**Theorem 3.6** (Verification). *Let  $\eta \in H_+$  and let  $x^*(\cdot)$  be a solution of (11) such that  $x^*(\cdot) > 0$ ; let  $c^*(\cdot)$  be the strategy defined by (12). Then  $c^*(\cdot)$  is admissible and optimal for the problem.*

**Proof.** As said above the fact that  $c^*(\cdot)$  is admissible is a direct consequence of the assumption  $x^*(\cdot) > 0$  and of the definition of  $c^*(\cdot)$ .

Set  $X^*(\cdot) := X(\cdot; \eta, c^*(\cdot))$  and let  $s > 0$ . Let  $p_1(s) \in L^2_{-T}$  be such that

$$(V_{\eta_0}(X^*(s)), p_1(s)) \in D^+V(X^*(s))$$

and let

$$\varphi(\zeta) := V(X^*(s)) + \langle (V_{\eta_0}(X^*(s)), p_1(s)), \zeta - X^*(s) \rangle, \quad \zeta \in H,$$

so that

$$\varphi(X^*(s)) = V(X^*(s)), \quad \varphi(\zeta) \geq V(\zeta), \quad \zeta \in H.$$

From Proposition 2.13 we know that  $\varphi \in \tau$ , so that

$$\begin{aligned} \liminf_{h \uparrow 0} \frac{e^{-\rho(s+h)} V(X^*(s+h)) - e^{-\rho s} V(X^*(s))}{h} &\geq \liminf_{h \uparrow 0} \frac{e^{-\rho(s+h)} \varphi(X^*(s+h)) - e^{-\rho s} \varphi(X^*(s))}{h} \\ &= e^{-\rho s} \left[ \mathcal{L}^{c^*(s)} \varphi \right] (X^*(s)) = e^{-\rho s} \left[ -\rho V(X^*(s)) + \langle X^*(s), A^*(V_{\eta_0}(X^*(s)), p_1(s)) \rangle \right. \\ &\quad \left. + f(X^*(s)) V_{\eta_0}(X^*(s)) + c^*(s) V_{\eta_0}(X^*(s)) \right]. \end{aligned}$$

Due to the definition of  $c^*(\cdot)$  we get

$$\begin{aligned} \liminf_{h \uparrow 0} \frac{e^{-\rho(s+h)} V(X^*(s+h)) - e^{-\rho s} V(X^*(s))}{h} &+ e^{-\rho s} [U_1(c^*(s)) + U_2(X_0^*(s))] \\ &\geq e^{-\rho s} \left[ -\rho V(X^*(s)) + \langle X^*(s), A^*(V_{\eta_0}(X^*(s)), p_1(s)) \rangle \right. \\ &\quad \left. + f(X^*(s)) V_{\eta_0}(X^*(s)) + \mathcal{H}(X^*(s)) + U_2(X_0^*(s)) \right]. \end{aligned}$$

Due to the subsolution property of  $V$  we get

$$\liminf_{h \uparrow 0} \frac{e^{-\rho(s+h)} V(X^*(s+h)) - e^{-\rho s} V(X^*(s))}{h} + e^{-\rho s} [U_1(c^*(s)) + U_2(X_0^*(s))] \geq 0.$$

The function  $s \mapsto e^{-\rho s} V(X^*(s))$  and the function  $s \mapsto e^{-\rho s} [U_1(c^*(s)) + U_2(X_0^*(s))]$  are continuous; therefore we can apply Lemma 3.4 on  $[0, M]$ ,  $M > 0$ , getting

$$e^{-\rho M} V(X^*(M)) + \int_0^M e^{-\rho s} [U_1(c^*(s)) + U_2(X_0^*(s))] ds \geq V(\eta).$$

Since  $V, U_1, U_2$  are bounded from above, taking the limsup for  $M \rightarrow +\infty$  we get by Fatou's Lemma

$$\int_0^{+\infty} e^{-\rho s} [U_1(c^*(s)) + U_2(X_0^*(s))] ds \geq V(\eta),$$

which gives the claim.  $\square$

**Remark 3.7.** We have given in Theorem 3.6 a sufficient condition of optimality: indeed, we have proved that if the feedback map defines an admissible strategy then such a strategy is optimal. Of course, a natural question arising is whether, at least with a special choice of data, such a condition is also necessary for the optimality, i.e. if, given an optimal strategy, it can be written as feedback of the associated optimal state. From the viscosity point of view the answer to this question relies in requiring that the value function is a *bilateral* viscosity subsolution of (8) along the optimal state trajectory, i.e. requiring that the value function satisfies the property of Definition 2.14-(i) also with the reverted inequality along this trajectory.

Such a property of the value function is related to the so-called *backward dynamic programming principle* which is, in turn, related to the backward study of the state equation (see [8],

Chapter III, Section 2.3). Differently from the finite-dimensional case, this topic is not standard in infinite-dimension unless the operator  $A$  is the generator of a strongly continuous *group*, which is not our case.

However, in our case we can use the delay original setting of the state equation to approach this topic. Then the problem reduces to find, at least for sufficient regular data, a *backward continuation* of the solution. This problem is faced, e.g., in [26], Chapter 2, Section 5. Unfortunately our equation does not fit the main assumption required therein, which in our setting basically corresponds to require that the function  $a(\cdot)$ , seen as measure, has an atom at  $-T$ . Investigation on this is left for future research. ■

### 3.1 The closed loop equation

Up to now we did not make any further assumption on the functions  $a$  and  $U_2$  beyond Hypotheses 2.2 and 2.5; in particular it could be  $U_2 \equiv 0$ . However without any further assumption we have no information on the behaviour of  $V_{\eta_0}$  when we approach the boundary of  $\mathcal{D}(V)$  and therefore we are not able to say anything about the existence of solutions of the closed loop equation and whether they satisfy or not the state constraint. So basically we cannot say whether the hypothesis of Theorem 3.6 is satisfied or not. In order to give sufficient conditions for that, we need to do some further assumptions.

**Hypothesis 3.8.** We will make use of the following assumptions

$$(i) \ U_2 \text{ is not integrable at } 0^+, \quad (ii) \ \int_{-\varepsilon}^0 a(\xi) d\xi > 0, \quad \forall \varepsilon > 0. \quad (16)$$

■

Also we need the following Lemma; we refer to [21] for the proof.

**Lemma 3.9.** *Let  $X(\cdot), \bar{X}(\cdot)$  be the mild solutions to (5) starting respectively from  $\eta, \bar{\eta} \in H$  and both under the null control. Then there exists a constant  $C > 0$  such that*

$$\|X(t) - \bar{X}(t)\|_{-1} \leq C \|\eta - \bar{\eta}\|_{-1}, \quad \forall t \in [0, T].$$

*In particular*

$$|X_0(t) - \bar{X}_0(t)| \leq rC \|\eta - \bar{\eta}\|_{-1}, \quad \forall t \in [0, T].$$

□

**Lemma 3.10.**

1. *The following holds*

$$\partial_{\|\cdot\|} \mathcal{D}(V) = \partial_{\|\cdot\|_{-1}} \mathcal{D}(V).$$

*Thanks to the previous equality we write without ambiguity  $\partial \mathcal{D}(V)$  for denoting the boundary of  $\mathcal{D}(V)$  referred to  $\|\cdot\|$  or  $\|\cdot\|_{-1}$  indifferently.*

2. *Suppose that (16)-(i) holds; then*

$$\lim_{\eta \rightarrow \bar{\eta}} V_{\eta_0}(\eta) = +\infty, \quad \forall \bar{\eta} \in \partial \mathcal{D}(V),$$

*where the limit is taken with respect to  $\|\cdot\|$ .*

**Proof.** We work with the original one-dimensional state equation with delay.

1. First of all note that, thanks to Proposition 2.8 and Proposition 2.12, the set  $\mathcal{D}(V)$  has the following structure

$$\mathcal{D}(V) = \bigcup_{\eta_1 \in L^2_{-T}} ((\eta_0^{\eta_1}, +\infty) \times \{\eta_1\}), \quad (17)$$

where, for  $\eta_1 \in L^2_{-T}$ , we set  $\eta_0^{\eta_1} = \inf\{\eta_0 > 0 \mid (\eta_0, \eta_1(\cdot)) \in \mathcal{D}(V)\}$ . For any  $\eta \in H$  set  $x^\eta(\cdot) := x(\cdot; \eta, 0)$  and consider the function  $g : H \rightarrow \mathbb{R}$  defined by

$$g(\eta_0, \eta_1(\cdot)) := \inf_{t \in [0, T]} x^\eta(t).$$

Thanks to Lemma 3.9 this function is continuous (with respect to both the norms  $\|\cdot\|$  and  $\|\cdot\|_{-1}$ ), so we have the following representation of  $\mathcal{D}(V)$  in terms of  $g$ :

$$\mathcal{D}(V) = \{g > 0\}.$$

Lemma 2.6 shows that  $g$  is increasing with respect to the first variable. Actually  $g$  is strictly increasing with respect to the first variable. Let us show this fact. Let  $\eta_1 \in L^2_{-T}$  and take  $\eta_0, \bar{\eta}_0 \in \mathbb{R}$  such that  $\eta_0 > \bar{\eta}_0$ . Define  $y(\cdot) := x(\cdot; (\eta_0, \eta_1(\cdot)), 0)$ ,  $\bar{x}(\cdot) := x(\cdot; (\bar{\eta}_0, \eta_1(\cdot)), 0)$  and let  $z(\cdot), \bar{z}(\cdot)$  be respectively the solutions on  $[0, T]$  of the differential problems *without delay*

$$\begin{cases} z'(t) = rz(t) + f_0 \left( z(t), \int_{-T}^0 a(\xi) x(t + \xi) d\xi \right), \\ z(0) = \eta_0, \end{cases}$$

$$\begin{cases} \bar{z}'(t) = r\bar{z}(t) + f_0 \left( \bar{z}(t), \int_{-T}^0 a(\xi) \bar{x}(t + \xi) d\xi \right), \\ \bar{z}(0) = \bar{\eta}_0, \end{cases}$$

Then we have, on the interval  $[0, T]$ ,  $\bar{z}(\cdot) \equiv x(\cdot)$  and, by comparison criterion,  $y(\cdot) \geq z(\cdot)$ ; moreover we can apply the classic Cauchy-Lipschitz Theorem for ODEs getting uniqueness for the solutions of the above ODEs, which yields  $z(\cdot) > \bar{z}(\cdot)$  on  $[0, T]$ . Thus  $y(\cdot) > x(\cdot)$  on  $[0, T]$ , proving that  $g$  is strictly increasing with respect to the first variable.

The continuity (with respect to both the norms) of  $g$ , (17) and the fact that  $g$  is strictly increasing with respect to the first variable lead to have

$$\partial_{\|\cdot\|} \mathcal{D}(V) = \partial_{\|\cdot\|_{-1}} \mathcal{D}(V) = \{g = 0\} = \bigcup_{\eta_1 \in L^2_{-T}} (\{\eta_0^{\eta_1}\} \times \{\eta_1\}).$$

2. We will intend the topological notions referred to  $\|\cdot\|$ . Firstly we prove that

$$\lim_{\eta \rightarrow \bar{\eta}} V(\eta) = -\infty, \quad \forall \bar{\eta} \in \partial \mathcal{D}(V).$$

Let  $\bar{\eta} \in \partial \mathcal{D}(V)$  and let  $(\eta^n) \subset \mathcal{D}(V)$  be a sequence such that  $\eta^n \rightarrow \bar{\eta}$ . We can suppose without loss of generality that  $(\eta^n) \subset B(\bar{\eta}, 1)$ . Set

$$x^n(\cdot) := x(\cdot; \eta^n, 0), \quad p^n := \sup_{\xi \in [0, 2T]} x^n(\xi).$$

Thanks to Lemma 3.9 there exists  $K > 0$  such that  $p^n \leq K$  for any  $n \in \mathbb{N}$ . So, since  $f_0(x, y) \leq C_0(1 + |x| + |y|)$  for some  $C_0 > 0$ , we have for the dynamics of  $x^n(\cdot)$  in the interval  $[0, 2T]$

$$\frac{d}{dt} x^n(t) \leq r x^n(t) + R,$$

where

$$R := C_0 \left( 1 + K + \|a\|_{L^2_{-T}} (\|\bar{\eta}_1\|_{L^2_{-T}} + 1) + \|a\|_{L^2_{-T}} T^{1/2} K \right).$$

Therefore there exists  $C > 0$  such that, for any  $s \in [0, T)$ ,  $n \in \mathbb{N}$ ,

$$x^n(t) \leq x^n(s) e^{r(t-s)} + \frac{R}{r} (e^{r(t-s)} - 1) \leq x^n(s)(1 + C(t-s)) + C(t-s), \quad t \in [s, 2T]. \quad (18)$$

By continuity of  $g$  we have  $\lim_{n \rightarrow \infty} g(\eta_0^n, \eta_1^n(\cdot)) = 0$ . Thus for any  $\varepsilon > 0$  we can find  $n_0 \in \mathbb{N}$  such that, for  $n \geq n_0$ , there exists  $s_n \in [0, T)$  such that

$$x^n(s_n) \leq \varepsilon. \quad (19)$$

We want to show that

$$\int_0^{+\infty} e^{-\rho t} U_2(x^n(t)) dt \longrightarrow -\infty, \quad n \rightarrow \infty. \quad (20)$$

For this purpose, since  $U_2$  is bounded from above, it is clear that we can assume without loss of generality  $U_2(\cdot) \leq 0$ . We have for  $n \geq n_0$ , taking into account (18) and (19),

$$\begin{aligned} \int_0^{+\infty} e^{-\rho t} U_2(x^n(t)) dt &\leq e^{-2\rho T} \int_{s_n}^{2T} U_2(x^n(s_n)(1 + C(t-s_n)) + C(t-s_n)) dt \\ &\leq e^{-2\rho T} \int_{s_n}^{2T} U_2(\varepsilon(1 + C(t-s_n)) + C(t-s_n)) dt \\ &\leq \frac{e^{-2\rho T}}{C(\varepsilon + 1)} \int_{\varepsilon}^{CT} U_2(x) dx. \end{aligned}$$

Therefore, by the arbitrariness of  $\varepsilon$  and since  $U_2$  is not integrable at  $0^+$ , we get (20). This is enough to conclude that  $J(\eta^n; 0) \rightarrow -\infty$ , as  $n \rightarrow \infty$ . Of course we have  $x^n(\cdot) \geq x(\cdot; \eta^n, c(\cdot))$  for any  $c(\cdot) \in \mathcal{C}(\eta^n)$ . Since  $U_1$  is bounded from above this is enough to say that also  $V(\eta^n) \rightarrow -\infty$ , as  $n \rightarrow \infty$ .

Now we prove the claim. Let  $\bar{\eta} \in \partial \mathcal{D}(V)$  and  $(\eta^n) \subset \mathcal{D}(V)$  be such that  $\eta^n \rightarrow \bar{\eta}$ , suppose without loss of generality  $(\eta_n) \subset B(\bar{\eta}, 1)$  and set  $x^n(\cdot) := x(\cdot; (\eta_0^n + 1, \eta_1^n), 0) > 0$ . Since  $f_0$  is Lipschitz continuous and nondecreasing on the second variable, there exists  $C > 0$  such that

$$f_0 \left( x(t), \int_{-T}^0 a(\xi) x(t + \xi) d\xi \right) \geq -C \left( 1 + x(t) + \|a\|_{L^2_{-T}} (\|\bar{\eta}_1\|_{L^2_{-T}} + 1) \right) =: -\tilde{R}.$$

Suppose  $\tilde{R} \leq 0$ . Then  $\frac{d}{dt} x^n(t) \geq r x^n(t)$ , so that, since  $\eta_0^n > 0$ , we have  $x^n(t) \geq \eta_0^n + 1 \geq 1$ . This leads to the estimate

$$V(\eta_0^n + 1, \eta_1^n(\cdot)) \geq K, \quad n \in \mathbb{N},$$

for some  $K > 0$ . Due to the concavity of  $V$  we have the estimate

$$V_{\eta_0}(\eta^n) \geq V(\eta_0^n + 1, \eta_1^n(\cdot)) - V(\eta_0^n, \eta_1^n(\cdot)) \geq K - V(\eta_0^n, \eta_1^n(\cdot)) \rightarrow +\infty,$$

i.e. the claim.

Suppose then  $\tilde{R} > 0$  and set  $x^n(\cdot) := x(\cdot; (\eta_0^n + \tilde{R}/r, \eta_1^n), 0)$ . Then  $\frac{d}{dt}x^n(t) \geq rx^n(t) - \tilde{R}$ , so that, since  $\eta_0^n > 0$ , we have  $x^n(t) \geq \tilde{R}/r > 0$ . This leads to the estimate

$$V(\eta_0^n + \tilde{R}/r, \eta_1^n(\cdot)) \geq K, \quad n \in \mathbb{N},$$

for some  $K > 0$ . Due to the concavity of  $V$  we have the estimate

$$V_{\eta_0}(\eta^n) \geq \frac{r}{\tilde{R}} \left[ V(\eta_0^n + \tilde{R}/r, \eta_1^n(\cdot)) - V(\eta_0^n, \eta_1^n(\cdot)) \right] \geq \frac{r}{\tilde{R}} [K - V(\eta_0^n, \eta_1^n(\cdot))] \rightarrow +\infty,$$

i.e. the claim.  $\square$

**Proposition 3.11.** *Let (16) hold, let  $\eta \in H_{++}$  and consider the closed-loop delay state equation (11). Then this equation admits a solution  $x^*(\cdot) \in C^1([0, +\infty); \mathbb{R})$ . Moreover, for all  $t \geq 0$ ,*

$$x^*(t) > 0, \quad (x^*(t), x^*(t + \xi)|_{\xi \in [-T, 0]}) \in \mathcal{D}(V).$$

*In particular the feedback strategy defined in (12) is admissible.*

**Proof.** Thanks to Lemma 3.10, if  $U_2$  is not integrable at  $0^+$  we can extend the map  $C$  to a continuous map defined on the whole space  $(H, \|\cdot\|)$  defining  $C \equiv 0$  on  $\mathcal{D}(V)^c$ . We set

$$G(\eta) := r\eta_0 + f(\eta) - C(\eta), \quad \eta \in H,$$

and note that  $G$  is continuous.

*Local existence.* Let  $\bar{\eta} \in H$  the initial datum for the equation. We have to show the local existence of a solution of

$$\begin{cases} x'(t) = G((x(t), x(t + \xi)|_{\xi \in [-T, 0]})) , \\ x(0) = \bar{\eta}_0, \quad x(s) = \bar{\eta}_1(s), \quad s \in [-T, 0), \end{cases}$$

Since  $G$  is continuous, there exists  $b > 0$  such that  $m := \sup_{\|\eta - \bar{\eta}\|^2 \leq b} |G(\eta)| < +\infty$ . By continuity of translations in  $L^2(\mathbb{R}; \mathbb{R})$  we can find  $a \in [0, T]$  such that

$$\int_{-T}^{-t} |\bar{\eta}_1(t + \xi) - \bar{\eta}_1(\xi)|^2 d\xi \leq b/4, \quad \forall t \in [0, a];$$

moreover, without loss of generality, we can suppose that  $\int_{-a}^0 |\bar{\eta}_1(\xi)|^2 d\xi \leq b/16$ . Set

$$\alpha := \min \left\{ a, \frac{b}{2m}, \frac{b}{16} (b + 2|\bar{\eta}_0|^2)^{-1} \right\}.$$

Define

$$M := \{x(\cdot) \in C([0, \alpha]; \mathbb{R}) \mid |x(\cdot) - \bar{\eta}_0|^2 \leq b/2\};$$

$M$  is a convex closed subset of the Banach space  $C([0, \alpha]; \mathbb{R})$  endowed with the sup-norm. Define

$$x(t + \xi) := \bar{\eta}_1(t + \xi), \quad \text{if } t + \xi \leq 0,$$

and observe that, for  $t \in [0, \alpha]$ ,  $x(\cdot) \in M$ ,

$$\begin{aligned} \int_{-t}^0 |x(t+\xi) - \bar{\eta}_1(\xi)|^2 d\xi &\leq \int_{-t}^0 (2|x(t+\xi)|^2 + 2|\bar{\eta}_1(\xi)|^2) d\xi \\ &\leq 2 \left[ \int_{-t}^0 (2(|x(t+\xi) - \bar{\eta}_0|)^2 + 2|\bar{\eta}_0|^2) d\xi + \int_{-t}^0 |\bar{\eta}_1(\xi)|^2 d\xi \right] \\ &\leq 2 \left[ 2t \left( \frac{b}{2} + |\bar{\eta}_0|^2 \right) + \frac{b}{16} \right] \leq b/4 \end{aligned}$$

So, for  $t \in [0, \alpha]$ ,  $x(\cdot) \in M$ , we have

$$\begin{aligned} \|(x(t), x(t+\xi)|_{\xi \in [-T, 0]}) - \bar{\eta}\|^2 &\leq |x(t) - \bar{\eta}_0|^2 + \int_{-t}^0 |x(t+\xi) - \bar{\eta}_1(\xi)|^2 d\xi + \int_{-T}^{-t} |\bar{\eta}_1(t+\xi) - \eta_1(\xi)|^2 d\xi \\ &\leq b/2 + b/4 + b/4 = b. \end{aligned}$$

Define, for  $t \in [0, \alpha]$ ,  $x(\cdot) \in M$ ,

$$[\mathcal{J}x](t) := \bar{\eta}_0 + \int_0^t G(x(s), x(s+\xi)|_{\xi \in [-T, 0]}) ds, \quad t \in [0, \alpha].$$

We have

$$\begin{aligned} |[\mathcal{J}x](t) - \eta_0| &\leq \int_0^t |G(x(s), x(s+\xi)|_{\xi \in [-T, 0]})| ds \\ &\leq tm \leq b/2. \end{aligned}$$

Therefore we have proved that  $\mathcal{J}$  maps the closed and convex set  $M$  in itself. We want to prove that  $\mathcal{J}$  admits a fixed point, i.e., by definition of  $\mathcal{J}$ , the solution we are looking for. By Schauder's Theorem it is enough to prove that  $\mathcal{J}$  is completely continuous, i.e. that  $\overline{\mathcal{J}(M)}$  is compact. For any  $x(\cdot) \in M$ , we have the estimate

$$|[\mathcal{J}x](t) - [\mathcal{J}x](\bar{t})| \leq \int_{t \wedge \bar{t}}^{t \vee \bar{t}} |G(x(s), x(s+\xi)|_{\xi \in [-T, 0]})| ds \leq m|t - \bar{t}|, \quad t, \bar{t} \in [0, \alpha].$$

Therefore  $\mathcal{J}(M)$  is a uniformly bounded and equicontinuous family in the space  $C([0, \alpha]; \mathbb{R})$ . Thus, by Ascoli-Arzelà Theorem,  $\overline{\mathcal{J}(M)}$  is compact.

*Global existence.* Let  $\eta \in H_{++}$  and let  $x^*(\cdot)$  be the solution of equation (11) defined on an interval  $[0, \beta)$ ,  $\beta > 0$ . Note that, by continuity of  $f_0$ ,  $C$ , we have  $x^*(\cdot) \in C^1([0, \beta); \mathbb{R})$ .

Since  $C(\cdot) \geq 0$ , we have  $x^*(\cdot) \leq x(\cdot; \eta, 0)$ ; therefore  $x^*(\cdot)$  is dominated from above on  $[0, \beta)$  by

$$\max_{t \in [0, \beta]} x(\cdot; \eta, 0).$$

We want to show that it is also dominated from below in order to apply the extension argument. Let us suppose that  $x^*(\bar{t}) = 0$  for some  $\bar{t} \in [0, \beta)$ . We want to show that this leads to a contradiction, so that, without loss of generality, we can suppose that

$$\bar{t} = \min\{t \in (0, \beta) \mid x^*(t) = 0\}.$$

Therefore  $x^*(\cdot) > 0$  in a left neighborhood of  $\bar{t}$ . Since  $f_0$  satisfies (2) and thanks to (16)-(ii), we must have  $\frac{d}{dt}x^*(\bar{t}) > 0$ , which contradicts  $x^*(\cdot) > 0$  in a left neighborhood of  $\bar{t}$ . Therefore we can say that  $x^*(\cdot) > 0$  on  $[0, \beta)$ , so that in particular  $x^*(\cdot)$  is bounded from below by 0 on  $[0, \beta)$ . Therefore, arguing as in the classical extension theorems for ODE, we could show that we can extend  $x^*(\cdot)$  to a solution defined on  $[0, +\infty)$  and, again by the same argument above, it will be  $x^*(\cdot) > 0$  on  $[0, +\infty)$ .  $\square$



## 4 Approximation results

In this section we obtain some approximation results which may be used in order to produce  $\varepsilon$ -optimal controls for a wider class of problems. Herein we assume that

$$rx + f_0(x, 0) \geq 0, \quad \forall x \geq 0, \quad (21)$$

which implies in particular

$$\exists \delta > 0 \text{ such that } rx + f_0(x, 0) \geq 0, \quad \forall x \in (0, \delta]. \quad (22)$$

In fact, all the results given below hold under (22) as well. We assume (21) only to simplify the proofs. Moreover we incorporate the term  $rx$  in the state equation within the term  $f_0$ , so that consistently with (21) we assume that

$$f_0(x, 0) \geq 0, \quad \forall x \geq 0. \quad (23)$$

Recall that, for a control problem, an  $\varepsilon$ -optimal strategy is a strategy  $\varepsilon$ -near to optimality. Precisely, in our problem

**Definition 4.1.** Let  $\eta \in \mathcal{D}(V)$ ,  $\varepsilon > 0$ ; an admissible control  $c^\varepsilon(\cdot) \in \mathcal{C}(\eta)$  is said  $\varepsilon$ -optimal for the initial state  $\eta$  if  $J(\eta; c^\varepsilon(\cdot)) > V(\eta) - \varepsilon$ .  $\square$

We will use the same concept for the control problems defined in the following.

### 4.1 The case without utility on the state

In the previous section we introduced an assumption of no integrability of the utility function  $U_2$ . This was necessary in order to ensure the existence of solutions for the closed loop equation and the admissibility of the feedback strategy. This fact is quite uncomfortable, because usually in consumption problems the objective functional is given by a utility depending only on the consumption variable, i.e. the case  $U_2 \equiv 0$  should be considered. Of course we could take a  $U_2$  heavily negative in a right neighborhood of 0 and equal to 0 out of this neighborhood, considering this as a forcing on the state constraint (states too near to 0 must be avoided). However we want to give here an approximation procedure to partly treat also the case  $U_2 \equiv 0$ , giving a way to construct at least  $\varepsilon$ -optimal strategies in this case.

So, let us consider a sequence of real functions  $(U_2^n)$  such that

$$U_2^n \uparrow 0, \quad U_2^n \text{ not integrable at } 0^+, \quad U_2^n \equiv 0 \text{ on } [1/n, +\infty). \quad (24)$$

Let us denote by  $J^n$  and  $V^n$  respectively the objective functionals and the value functions of the problems where the utility on the state is given by  $U_2^n$  and by  $J^0$  and  $V^0$  respectively the objective functional and the value function of the problem where the utility on the state disappears, i.e.  $U_2 \equiv 0$ . It is immediate to see that monotonicity implies

$$V^n \uparrow g \leq V^0. \quad (25)$$

Thanks to the previous section, for any problem  $V^n$ ,  $n \in \mathbb{N}$ , we have an optimal feedback strategy  $c_n^*(\cdot)$ .

**Lemma 4.2.** *Let  $\eta \in \mathcal{D}(V^0) \subset H_+$ . Then, for any  $\varepsilon > 0$ , there exists an  $\varepsilon$ -optimal strategy  $c^\varepsilon(\cdot) \in \mathcal{C}(\eta)$  for  $V^0(\eta)$  such that*

$$\inf_{t \in [0, +\infty)} x(t; \eta, c^\varepsilon(\cdot)) > 0.$$

**Proof.** Let  $\varepsilon > 0$  and take an  $\varepsilon/2$ -optimal control  $c^{\varepsilon/2}(\cdot) \in \mathcal{C}(\eta)$  for  $V^0(\eta)$ . Let  $M > T$  be such that

$$\frac{1}{\rho} e^{-\rho M} (\bar{U}_1 - U_1(0)) < \varepsilon/2. \quad (26)$$

Define the control

$$c^\varepsilon(t) := \begin{cases} c^{\varepsilon/2}(t), & \text{for } t \in [0, M], \\ 0, & \text{for } t > M. \end{cases}$$

By Lemma 2.6 we have

$$x(\cdot; \eta, c^\varepsilon(\cdot)) \geq x(\cdot; \eta, c^{\varepsilon/2}(\cdot))$$

and, by the assumption (23) and since  $c^\varepsilon(t) = 0$  for  $t \geq M$ , it is not difficult to see that

$$x(t; \eta, c^\varepsilon(\cdot)) \geq x(M; \eta, c^\varepsilon(\cdot)), \quad \text{for } t \geq M,$$

so that

$$\inf_{t \in [0, +\infty)} x(t; \eta, c^\varepsilon(\cdot)) = \inf_{t \in [0, M]} x(t; \eta, c^\varepsilon(\cdot)) > 0.$$

We claim that  $c^\varepsilon(\cdot)$  is  $\varepsilon$ -optimal for  $V^0(\eta)$ , which yields the claim. Since  $c^{\varepsilon/2}(\cdot)$  is  $\varepsilon/2$ -optimal for  $V^0(\eta)$ , taking also into account (26), we get

$$\begin{aligned} V^0(\eta) - \int_0^{+\infty} e^{-\rho t} U_1(c^\varepsilon(t)) dt &= V^0(\eta) - \int_0^{+\infty} e^{-\rho t} U_1(c^{\varepsilon/2}(t)) dt \\ &\quad + \int_M^{+\infty} e^{-\rho t} (U_1(c^{\varepsilon/2}(t)) - U_1(0)) dt < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

□

**Proposition 4.3.** *Let  $\eta \in \mathcal{D}(V^0)$  and  $\varepsilon > 0$ . Then  $V^n(\eta) \rightarrow V^0(\eta)$  and, when  $n$  is large enough,  $c_n^*(\cdot)$  is  $\varepsilon$ -optimal for  $V^0(\eta)$ .*

**Proof.** Let  $\varepsilon > 0$  and take an  $\varepsilon$ -optimal control  $c^\varepsilon(\cdot) \in \mathcal{C}(\eta)$  for  $V^0(\eta)$  such that (Lemma 4.2)

$$m := \inf_{t \in [0, +\infty)} x(t; \eta; c^\varepsilon(\cdot)) > 0.$$

Take  $n \in \mathbb{N}$  such that  $1/n < m$ . Since  $U_2^n \equiv 0$  on  $[m, +\infty)$ , we have

$$\begin{aligned} V^0(\eta) - \varepsilon &\leq J(\eta; c^\varepsilon(\cdot)) = \int_0^{+\infty} e^{-\rho t} U_1(c^\varepsilon(t)) dt \\ &= \int_0^{+\infty} e^{-\rho t} [U_1(c^\varepsilon(t)) + U_2^n(x(t; \eta, c^\varepsilon(\cdot)))] dt \\ &= J^n(\eta, c^\varepsilon(\cdot)) \leq V^n(\eta) = J^n(\eta, c_n^*(\cdot)) \leq J^0(\eta, c_n^*(\cdot)). \end{aligned}$$

The latter inequality, together with (25), proves both the claims. □

## 4.2 The case with pointwise delay in the state equation

In this subsection we want to show that our problem is a good approximation for growth models with time to build and concentrated lag and discuss why our approach cannot work directly when the delay is concentrated in a point. In this case the state equation is

$$\begin{cases} y'(t) = f_0(y(t), y(t - \frac{T}{2})) - c(t), \\ y(0) = \eta_0, \quad y(s) = \eta_1(s), \quad s \in [-T, 0). \end{cases} \quad (27)$$

It is possible to prove, as done in Theorem 2.4, that this equation admits, for every  $\eta \in H_+$ , and for every  $c(\cdot) \in L^1_{loc}([0, +\infty); \mathbb{R})$ , a unique absolutely continuous solution. We denote this solution by  $y(\cdot; \eta, c(\cdot))$ . The aim is to maximize, over the set

$$\mathcal{C}_{ad}^0(\eta) := \{c(\cdot) \in L^1_{loc}([0, +\infty); \mathbb{R}) \mid y(\cdot; \eta, c(\cdot)) > 0\}, \quad (28)$$

the functional

$$J_0(\eta, c(\cdot)) = \int_0^{+\infty} e^{-\rho t} [U_1(c(\cdot)) + U_2(y(t; \eta, c(\cdot)))] dt.$$

Denote by  $V_0$  the associated value function. By monotonicity of  $f_0$  we straightly get  $H_{++} \subset \mathcal{D}(V_0)$ .

Let us take a sequence  $(a_k)_{k \in \mathbb{N}} \subset W_{-T}^{1,2}$  such that

$$a_k(-T) = 0, \quad \|a_k\|_{L^2_{-T}} \leq 1, \quad (16)\text{-(ii) holds true } \forall a_k, \quad a_k \xrightarrow{*} \delta_{-T/2} \text{ in } (C([-T, 0]; \mathbb{R}))^*, \quad (29)$$

where  $\delta_{-T/2}$  is the Dirac measure concentrated at  $-T/2$ . We denote by  $x_k(\cdot; \eta, c(\cdot))$  the unique solution of (1) where  $a(\cdot)$  is replaced by  $a_k(\cdot)$ .

**Proposition 4.4.** *Let  $\eta \in H_+$ ,  $c(\cdot) \in L^1_{loc}([0, +\infty); \mathbb{R})$  and set  $y(\cdot) := y(\cdot; \eta, c(\cdot))$ ,  $x_k(\cdot) := x_k(\cdot; \eta, c(\cdot))$ . Then there exists a continuous and increasing function  $h$  such that  $h(0) = 0$  and*

$$\sup_{s \in [0, t]} |x_k(s) - y(s)| \leq h(t) u_k(t), \quad t \in [0, +\infty), \quad (30)$$

where  $u_k(t) \rightarrow 0$ , as  $k \rightarrow \infty$ , uniformly on bounded sets.

**Proof.** Note that

$$\|a_k\|_{(C([-T, 0]; \mathbb{R}))^*} = \sup_{\|f\|_\infty=1} \left| \int_{-T}^0 a_k(\xi) f(\xi) d\xi \right| \leq \int_{-T}^0 |a_k(\xi)| d\xi \leq \|a_k\|_{L^2_{-T}} \cdot T^{1/2} \leq T^{1/2}. \quad (31)$$

Let  $t \geq 0$ ; we have, for any  $\zeta \in [0, t]$ ,

$$\begin{aligned} |x_k(\zeta) - y(\zeta)| &= \int_0^\zeta \left[ f_0 \left( x_k(s), \int_{-T}^0 a_k(\xi) x_k(s + \xi) d\xi \right) - f_0 \left( y(s), y \left( s - \frac{T}{2} \right) \right) \right] ds \\ &\leq C_{f_0} \left[ \int_0^t |x_k(s) - y(s)| ds + \int_0^t \left| \int_{-T}^0 a_k(\xi) (x_k(s + \xi) - y(s + \xi)) d\xi \right| ds \right. \\ &\quad \left. + \int_0^t \left| \int_{-T}^0 a_k(\xi) y(s + \xi) d\xi - y \left( s - \frac{T}{2} \right) \right| ds \right]. \end{aligned} \quad (32)$$

Call

$$g_k(t) := \sup_{s \in [-T, t]} |x_k(s) - y(s)| = \sup_{s \in [0, t]} |x_k(s) - y(s)|,$$

and set, for  $s \in [0, t]$ ,

$$u_k(s) := C_{f_0} \int_0^s \left| \int_{-T}^0 a_k(\xi) y(r + \xi) d\xi - y\left(r - \frac{T}{2}\right) \right| dr.$$

Note that, for every  $s \in [0, t]$ , the function  $[-T, 0] \ni \xi \mapsto x_k(s + \xi) - y(s + \xi)$  is continuous, therefore thanks to (31) we can write from (32)

$$g_k(t) \leq C_{f_0} \left[ \int_0^t g_k(s) ds + T^{1/2} \int_0^t g_k(s) ds + u_k(t) \right].$$

Therefore, setting  $K := C_{f_0}(1 + T^{1/2})$ , we get by Gronwall's Lemma

$$g_k(t) \leq u_k(t) + Kte^{Kt}u_k(t) =: h(t)u_k(t). \quad (33)$$

Note that, since  $a_k \xrightarrow{*} \delta_{-T/2}$  in  $(C([-T, 0]; \mathbb{R}))^*$ , we have the pointwise convergence

$$\int_{-T}^0 a_k(\xi) y(s + \xi) d\xi \longrightarrow y\left(s - \frac{T}{2}\right), \quad s \in [0, t];$$

moreover

$$\left| \int_{-T}^0 a_k(\xi) y(s + \xi) d\xi \right| \leq \|a_k\|_{L^2_{-T}} \cdot \|y(s + \xi)|_{\xi \in [-T, 0]}\|_{L^2_{-T}} \leq C_{\eta, c(\cdot)} < +\infty, \quad \forall s \in [0, t],$$

where the last inequality follows from the fact that the function  $[0, t] \rightarrow L^2_{-T}$ ,  $s \mapsto y(s + \xi)|_{\xi \in [-T, 0]}$  is continuous. Therefore we have by dominated convergence  $u_k(t) \rightarrow 0$ . By (33) we get (30).  $\square$

For  $k \in \mathbb{N}$ ,  $\eta \in H_+$ , let

$$\mathcal{C}_{ad}^k(\eta) := \{c(\cdot) \in L^1_{loc}([0, +\infty); \mathbb{R}) \mid x_k(\cdot; \eta, c(\cdot)) > 0\}, \quad (34)$$

Consider the problem of maximizing over  $\mathcal{C}_{ad}^k(\eta)$  the functional

$$J_k(\eta, c(\cdot)) := \int_0^{+\infty} e^{-\rho t} [U_1(c(t)) + U_2(x_k(t; \eta, c(\cdot)))] dt$$

and denote by  $V_k$  the associated value function. Note that, since we have assumed (23), straightly we get  $H_{++} \subset \mathcal{D}(V_k)$  for every  $k \in \mathbb{N}$ . Thanks to the previous section we have a sequence of optimal feedback strategies for the sequence of problems  $(V_k(\eta))_{k \in \mathbb{N}}$ , in the sense that we have a sequence  $(c_k^*(\cdot))_{k \in \mathbb{N}}$  of feedback controls such that  $c_k^*(\cdot) \in \mathcal{C}_{ad}^k(\eta)$  for every  $k \in \mathbb{N}$  and

$$J_k(\eta; c_k^*(\cdot)) = \sup_{c(\cdot) \in \mathcal{C}_{ad}^k(\eta)} J_k(\eta; c(\cdot)) =: V_k(\eta), \quad \forall k \in \mathbb{N}.$$

**Lemma 4.5.** *Let  $\eta \in H_{++}$ .*

- *For any  $\varepsilon > 0$  there exists an  $\varepsilon$ -optimal strategy  $c^\varepsilon(\cdot) \in \mathcal{C}_{ad}^0(\eta)$  for the problem  $V_0(\eta)$  such that*

$$\inf_{t \in [0, +\infty)} y(t; \eta, c^\varepsilon(\cdot)) > 0.$$

- Assume that

$$\lim_{x \rightarrow 0^+} [x U_2(x)] = -\infty. \quad (35)$$

Then, for any  $\varepsilon > 0$  there exists  $\nu > 0$  such that for any  $k \in \mathbb{N}$  there exists an  $\varepsilon$ -optimal control  $c_k^\varepsilon(\cdot) \in \mathcal{C}_{ad}^k(\eta)$  for the problem  $V_k(\eta)$  such that

$$\inf_{t \in [0, +\infty)} x_k(t; \eta, c_k^\varepsilon(\cdot)) \geq \nu. \quad (36)$$

**Proof.** (i) Let  $\varepsilon > 0$  and take an  $\varepsilon/2$ -optimal control  $c^{\varepsilon/2}(\cdot) \in \mathcal{C}_{ad}^0(\eta)$  for the problem  $V_0(\eta)$ . Take  $M > 0$  large enough to satisfy

$$\frac{1}{\rho} e^{-\rho M} (\bar{U}_1 - U_1(0)) < \varepsilon/2. \quad (37)$$

Define the control

$$c^\varepsilon(t) := \begin{cases} c^{\varepsilon/2}(t), & \text{for } t \in [0, M], \\ 0, & \text{for } t > M. \end{cases}$$

A comparison criterion like the one proved in Lemma 2.6 can be proved also for equation (27). Therefore we have

$$y(\cdot; \eta, c^\varepsilon(\cdot)) \geq y(\cdot; \eta, c^{\varepsilon/2}(\cdot)) \quad (38)$$

and, since we have assumed (23),

$$\inf_{t \in [0, +\infty)} y(t; \eta, c^\varepsilon(\cdot)) = \inf_{t \in [0, M]} y(t; \eta, c^{\varepsilon/2}(\cdot)).$$

We claim that  $y(\cdot; \eta, c^\varepsilon(\cdot))$  is  $\varepsilon$ -optimal for  $V_0$ , which yields the claim. Since  $c^{\varepsilon/2}(\cdot)$  is  $\varepsilon/2$ -optimal for  $V_0(\eta)$ , taking also into account (37) and (38),

$$\begin{aligned} V_0(\eta) - \int_0^{+\infty} e^{-\rho t} [U_1(c^\varepsilon(t)) + U_2(y(t; \eta, c^\varepsilon(\cdot)))] dt \\ = V_0(\eta) - \int_0^{+\infty} e^{-\rho t} (U_1(c^{\varepsilon/2}(t)) + U_2(y(t; \eta, c^{\varepsilon/2}(\cdot)))) dt \\ + \int_M^{+\infty} e^{-\rho t} (U_1(c^{\varepsilon/2}(t)) - U_1(0)) dt + \int_M^{+\infty} e^{-\rho t} [U_2(y(t; \eta, c^{\varepsilon/2}(\cdot)) - U_2(y(t; \eta, c^\varepsilon(t))))] dt \\ < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

(ii) Due to (23) we have  $x_k(\cdot; \eta, 0) \geq \eta_0$  for every  $k \in \mathbb{N}$ . Let

$$j_0 := \frac{U_1(0) + U_2(\eta_0)}{\rho}.$$

Then we have  $V_k(\eta) \geq J_k(\eta, 0) \geq j_0$  for every  $k \in \mathbb{N}$ . Take  $M > 0$  large enough to satisfy

$$\frac{1}{\rho} e^{-\rho M} (\bar{U}_1 - U_1(0)) < \varepsilon. \quad (39)$$

Arguing as done to get (18) and taking into account the comparison criterion, we can find  $C_M > 0$  such that, for every  $k \in \mathbb{N}$  and for every  $c(\cdot) \in \mathcal{C}_{ad}^k(\eta)$ , we have for all  $s \in [0, M]$  and for all  $t \in [s, M+1]$ ,

$$x_k(t; \eta, c(\cdot)) \leq x_k(s; \eta, c(\cdot))(1 + C_M(t-s)) + C_M(t-s). \quad (40)$$

Now take  $\nu > 0$  small enough to have

$$(i) \quad \nu < 1, \quad (ii) \quad \frac{\nu}{2C_M} < 1, \quad (iii) \quad \frac{\nu}{2C_M} U_2(2\nu) e^{-\rho(M+1)} < j_0 - \frac{\bar{U}_1 + \bar{U}_2}{\rho} - 1 < 0. \quad (41)$$

For  $k \in \mathbb{N}$ , thanks to the previous section we have optimal strategies in feedback form  $c_k^*(\cdot) \in \mathcal{C}_{ad}^k(\eta)$  for  $V_k$ ; we claim that  $x_k(t; \eta, c_k^*(\cdot)) > \nu$  for  $t \in [0, M]$  for every  $k \in \mathbb{N}$ . Indeed suppose by contradiction that for some  $t_0 \in [0, M]$  we have  $x_k(t_0; \eta, c_k^*(\cdot)) = \nu$ ; then by (40) and (41)-(i),(ii) we get that

$$x_k(t; \eta, c_k^*(\cdot)) \leq 2\nu, \quad \text{for } t \in \left[ t_0, t_0 + \frac{\nu}{2C_M} \right].$$

Therefore, by (41)-(iii),

$$\int_{t_0}^{t_0 + \frac{\nu}{2C_M}} e^{-\rho t} U_2^n(x_k(t; \eta, c_k^*(\cdot))) dt \leq j_0 - \frac{\bar{U}_1 + \bar{U}_2}{\rho} - 1.$$

This shows that

$$J_k(\eta, c_k^*(\cdot)) \leq j_0 - 1 \leq V_k(\eta) - 1.$$

This fact contradicts the optimality of  $c_k^*(\cdot)$ . Therefore we have proved that for the choice of  $\nu$  given by (41) we have

$$x_k(t; \eta, c_k^*(\cdot)) > \nu, \quad \text{for } t \in [0, M].$$

We can continue the strategy  $c_k^*(\cdot)$  after  $M$  taking the null strategy, i.e. defining the strategy

$$c_k^\varepsilon(\cdot) := \begin{cases} c_k^*(t), & \text{for } t \in [0, M], \\ 0, & \text{for } t > M. \end{cases} \quad (42)$$

Then by (23) we have  $x_k(\cdot; \eta, c_k^\varepsilon(\cdot)) > \nu$  for every  $k \in \mathbb{N}$ . We claim that  $c_k^\varepsilon(\cdot)$  is  $\varepsilon$ -optimal for  $V_k(\eta)$  for every  $k \in \mathbb{N}$ , which proves the claim. Indeed, taking into account the comparison criterion and (39) for the inequality in the following,

$$\begin{aligned} V_k(\eta) - \int_0^{+\infty} e^{-\rho t} (U_1(c_k^\varepsilon(t)) + U_2(x_k(t; \eta, c_k^\varepsilon(\cdot)))) dt \\ = V_k(\eta) - \int_0^{+\infty} e^{-\rho t} (U_1(c_k^*(t)) + U_2(x_k(t; \eta, c_k^*(\cdot)))) dt \\ + \int_M^{+\infty} e^{-\rho t} (U_1(c_k^*(t)) - U_1(0)) dt + \int_M^{+\infty} e^{-\rho t} (U_2(x_k(t; \eta, c_k^*(\cdot)) - U_2(x_k(t; \eta, c_k^\varepsilon(t)))) dt < \varepsilon. \end{aligned}$$

□

**Proposition 4.6.** *Let  $\eta \in H_{++}$  and suppose that (35) holds true. We have  $V_k(\eta) \rightarrow V_0(\eta)$ , as  $k \rightarrow \infty$ . Moreover for every  $\varepsilon > 0$  we can find a constant  $M_\varepsilon$  and a  $k_\varepsilon$  such that the strategy  $(c_{k_\varepsilon}^*)$  is the optimal feedback strategy for the problem of  $V_k$*

$$c_{k_\varepsilon, M_\varepsilon}(t) := \begin{cases} c_{k_\varepsilon}^*(t), & \text{for } t \in [0, M_\varepsilon], \\ 0, & \text{for } t > M_\varepsilon. \end{cases} \quad (43)$$

is  $\varepsilon$ -optimal strategy for the problem  $V_0(\eta)$ .

**Proof.** (i) Here we show that

$$\liminf_{k \rightarrow \infty} V_k(\eta) \geq V_0(\eta), \quad (44)$$

Let  $\varepsilon > 0$  and let  $c^\varepsilon(\cdot) \in \mathcal{C}_{ad}^0(\eta)$  be an  $\varepsilon$ -optimal strategy for the problem  $V_0(\eta)$ . Thanks to Lemma 4.5-(i) we can suppose without loss of generality  $2\nu_1 := \inf_{t \in [0, +\infty)} y(t; \eta, c^\varepsilon(\cdot)) > 0$ . The function  $U_1$  is uniformly continuous on  $[0, +\infty)$  and the function  $U_2$  is uniformly continuous on  $[\nu_1, +\infty)$ . Let  $\omega_{\nu_1}$  be a modulus of uniform continuity for both these functions. Take  $M > 0$  such that

$$-\frac{1}{\rho}e^{-\rho M}(\bar{U}_1 + \bar{U}_2) - \frac{1 - e^{-\rho M}}{\rho}\omega_{\nu_1}(\nu_1) + \frac{1}{\rho}e^{-\rho M}(U_1(0) + U_2(\nu_1)) \geq -\varepsilon, \quad (45)$$

Define

$$c_M^\varepsilon(t) := \begin{cases} c^\varepsilon(t), & \text{for } t \in [0, M], \\ 0, & \text{for } t > M. \end{cases}$$

Let  $k_M$  be such that

$$h(M)u_k(M) < \nu_1, \quad \forall k \geq k_M, \quad (46)$$

where  $u_k$  and  $h$  are the functions appearing in (30). Then, thanks to Proposition 4.4 and to the monotonicity property of  $f_0$ , it is straightforward to see that  $x_k(t; \eta, c_M^\varepsilon(\cdot)) \geq \nu_1 > 0$ , so that in particular  $c_M^\varepsilon(\cdot) \in \mathcal{C}_{ad}^k(\eta)$  for all  $k \geq k_M$ . For all  $k \geq k_M$ , we have, thanks to Proposition 4.4 and by definition of  $k_M$ ,

$$\begin{aligned} & \int_0^{+\infty} e^{-\rho t} [U_1(c_M^\varepsilon(t)) + U_2(x_k(t; \eta, c_M^\varepsilon(\cdot)))] dt \\ &= \int_0^M e^{-\rho t} [U_1(c_M^\varepsilon(t)) + U_2(x_k(t; \eta, c_M^\varepsilon(\cdot)))] dt + \int_M^{+\infty} e^{-\rho t} [U_1(c_M^\varepsilon(t)) + U_2(x_k(t; \eta, c_M^\varepsilon(\cdot)))] dt \\ &\geq \int_0^M e^{-\rho t} [U_1(c^\varepsilon(t)) + U_2(y(t; \eta, c^\varepsilon(\cdot)))] dt - \frac{1 - e^{-\rho M}}{\rho}\omega_{\nu_1}(\nu_1) + \frac{1}{\rho}e^{-\rho M}(U_1(0) + U_2(\nu_1)) \\ &\geq \int_0^{+\infty} e^{-\rho t} [U_1(c^\varepsilon(t)) + U_2(y(t; \eta, c^\varepsilon(\cdot)))] dt \\ &\quad - \frac{1}{\rho}e^{-\rho M}(\bar{U}_1 + \bar{U}_2) - \frac{1 - e^{-\rho M}}{\rho}\omega_{\nu_1}(\nu_1) + \frac{1}{\rho}e^{-\rho M}(U_1(0) + U_2(\nu_1)). \end{aligned}$$

so that by (45)

$$V_k(\eta) \geq V_0(\eta) - 2\varepsilon, \quad (47)$$

which shows (44).

(ii) Now we show that

$$\limsup_{k \rightarrow \infty} V_k(\eta) \leq V_0(\eta). \quad (48)$$

Let  $\varepsilon > 0$ ; thanks to Lemma 4.5-(ii) we can construct a sequence  $(c_k^\varepsilon(\cdot))_{k \in \mathbb{N}}$ ,  $c_k^\varepsilon(\cdot) \in \mathcal{C}_{ad}^k(\eta)$  for every  $k \in \mathbb{N}$ , of  $\varepsilon$ -optimal controls for the sequence of problems  $(V_k(\eta))_{k \in \mathbb{N}}$  such that

$$2\nu_2 := \inf_{k \in \mathbb{N}} \inf_{t \in [0, +\infty)} x_k(t; \eta, c_k^\varepsilon(\cdot)) > 0.$$

Let  $\omega_{\nu_2}$  be a modulus of uniform continuity for  $U_1$  on  $[0, +\infty)$  and for  $U_2$  on  $[\nu_2, +\infty)$ . Take  $\tilde{M} > 0$  such that

$$-\frac{1}{\rho}e^{-\rho\tilde{M}}(\bar{U}_1 + \bar{U}_2) - \frac{1 - e^{-\rho\tilde{M}}}{\rho}\omega_{\nu_2}(\nu_2) + \frac{1}{\rho}e^{-\rho\tilde{M}}(U_1(0) + U_2(\nu_2)) > -\varepsilon \quad (49)$$

and define the controls

$$c_{k,\tilde{M}}^\varepsilon(t) := \begin{cases} c_k^\varepsilon(t), & \text{for } t \in [0, \tilde{M}], \\ 0, & \text{for } t > \tilde{M}. \end{cases} \quad (50)$$

As before we can find  $k_{\tilde{M}}$  such that we have

$$h(\tilde{M})u_k(\tilde{M}) < \nu_2, \quad \forall k \geq k_{\tilde{M}}, \quad (51)$$

In this case we have  $y(\cdot; \eta, c_k^\varepsilon(\cdot)) \geq \nu_2$  and

$$\begin{aligned} & \int_0^{+\infty} e^{-\rho t} [U_1(c_{k,\tilde{M}}^\varepsilon(t)) + U_2(y(t; \eta, c_{k,\tilde{M}}^\varepsilon(\cdot)))] dt \\ &= \int_0^{\tilde{M}} e^{-\rho t} [U_1(c_{k,\tilde{M}}^\varepsilon(t)) + U_2(y(t; \eta, c_{k,\tilde{M}}^\varepsilon(\cdot)))] dt + \int_{\tilde{M}}^{+\infty} e^{-\rho t} [U_1(c_{k,\tilde{M}}^\varepsilon(t)) + U_2(y(t; \eta, c_{k,\tilde{M}}^\varepsilon(\cdot)))] dt \\ &\geq \int_0^{\tilde{M}} e^{-\rho t} [U_1(c_k^\varepsilon(t)) + U_2(x_k(t; \eta, c_k^\varepsilon(\cdot)))] dt - \frac{1 - e^{-\rho\tilde{M}}}{\rho}\omega_{\nu_2}(\nu_2) + \frac{1}{\rho}e^{-\rho\tilde{M}}(U_1(0) + U_2(\nu_2)) \\ &\geq \int_0^{+\infty} e^{-\rho t} [U_1(c_k^\varepsilon(t)) + U_2(x_k(t; \eta, c_k^\varepsilon(\cdot)))] dt \\ &\quad - \frac{1}{\rho}e^{-\rho\tilde{M}}(\bar{U}_1 + \bar{U}_2) - \frac{1 - e^{-\rho\tilde{M}}}{\rho}\omega_{\nu_2}(\nu_2) + \frac{1}{\rho}e^{-\rho\tilde{M}}(U_1(0) + U_2(\nu_2)). \quad (52) \end{aligned}$$

By (49), we get, for  $k \geq k_{\tilde{M}}$ ,

$$V_0(\eta) \geq V_k(\eta) - 2\varepsilon,$$

which proves (48).

(iii) The procedure of construction of  $c_{k,M}^\varepsilon$  in (ii) yields  $\varepsilon$ -optimal controls for the limit problem  $V_0(\eta)$ . Indeed, starting from  $\varepsilon > 0$ , we can compute  $\nu_1, \nu_2, M, \tilde{M}$  depending on  $\varepsilon$  such that (45) and (49) hold true. Then, if  $(a_k)_{k \in \mathbb{N}}$  is chosen in a clever way, for example if  $(a_k)_{k \in \mathbb{N}}$  is a sequence of gaussian densities, we can compute  $k_M, k_{\tilde{M}}$  such that (46)-(51) hold true. Thanks to (52) and (47), for every  $k \geq k_M \vee k_{\tilde{M}}$  the controls  $c_{k,\tilde{M}}^\varepsilon(\cdot)$  defined in (50) are  $4\varepsilon$ -optimal for the limit problem  $V^0(\eta)$ . Replacing  $\varepsilon$  with  $\varepsilon/4$  we get the controls in (43).  $\square$

**Remark 4.7.** When the delay is concentrated in a point in a linear way, we could tempted to insert the delay term in the infinitesimal generator  $A$  and try to work as done in Section 2.2. Unfortunately this is not possible. Indeed consider this simple case:

$$\begin{cases} y'(t) = ry(t) + y(t - T), \\ y(0) = \eta_0, \quad y(s) = \eta_1(s), \quad s \in [-T, 0), \end{cases}$$

In this case we can define

$$A : \mathcal{D}(A) \subset H \longrightarrow H, \quad (\eta_0, \eta_1(\cdot)) \longmapsto (r\eta_0 + \eta_1(-T), \eta_1'(\cdot)).$$



where again

$$\mathcal{D}(A) := \{\eta \in H \mid \eta_1(\cdot) \in W^{1,2}([-T, 0]; \mathbb{R}), \eta_1(0) = \eta_0\}.$$

The inverse of  $A$  is the operator

$$A^{-1} : (H, \|\cdot\|) \longrightarrow (\mathcal{D}(A), \|\cdot\|) \quad (\eta_0, \eta_1(\cdot)) \longmapsto \left( \frac{\eta_0 - c}{r}, c + \int_{-T}^{\cdot} \eta_1(\xi) d\xi \right),$$

where

$$c = \frac{1}{r+1} \eta_0 - \frac{r}{r+1} \int_{-T}^0 \eta_1(\xi) d\xi.$$

In this case we would have the first part of Lemma 3.9, but not the second part, because it is not possible to control  $|\eta_0|$  by  $\|\eta\|_{-1}$ . Indeed take for example  $r$  such that  $\frac{1-r}{1+r} = \frac{1}{2}$ , and  $(\eta^n)_{n \in \mathbb{N}} \subset H$  such that

$$\eta_0^n = 1/2, \quad \int_{-T}^0 \eta_1^n(\xi) d\xi = 1, \quad n \in \mathbb{N}.$$

We would have  $c = 1/2$ , so that  $\left| \frac{\eta_0^n - c}{r} \right| = 0$ . Moreover we can choose  $\eta_1^n$  such that, when  $n \rightarrow \infty$ ,

$$\int_{-T}^0 \left| \frac{1}{2} + \int_{-T}^s \eta_1^n(\xi) d\xi \right|^2 ds \longrightarrow 0.$$

Therefore we would have  $|\eta_0^n| = 1/2$  and  $\|\eta^n\|_{-1} \rightarrow 0$ . This shows that the second part of Lemma 3.9 does not hold. Once this part does not hold, then everything in the following argument breaks down. ■

### 4.3 The case with pointwise delay in the state equation and without utility on the state

Now we want to approximate the problem of optimizing, for  $\eta \in H_{++}$ ,

$$J_0^0(\eta, c(\cdot)) := \int_0^{+\infty} e^{-\rho t} U_1(c(t)) dt,$$

over the set (28), where  $y(\cdot; \eta, c(\cdot))$  follows the dynamics given by (27). Let us denote by  $V_0^0$  the corresponding value function and let us take a sequence of real functions  $(U_2^n)$  as in (24), but with the assumption of no integrability at  $0^+$  replaced by the stronger assumption

$$\lim_{x \rightarrow 0^+} x U_2^n(x) = -\infty, \quad \forall n \in \mathbb{N}.$$

Fix  $n \in \mathbb{N}$  and consider the sequence of functions  $(a_k)_{k \in \mathbb{N}}$  defined in (29). For  $k \in \mathbb{N}$ , consider the problem of maximizing over the set  $\mathcal{C}_{ad}^k(\eta)$  defined in (34) the functional

$$J_k^n(\eta, c(\cdot)) := \int_0^{+\infty} e^{-\rho t} (U_1(c(t)) + U_2^n(x_k(t; \eta, c(\cdot)))) dt,$$

where  $x_k(\cdot; \eta, c(\cdot))$  follows the dynamics given by (1) when  $a(\cdot)$  is replaced by  $a_k(\cdot)$ , and denote by  $V_k^n$  the associated value function.

Moreover, for  $k \in \mathbb{N}$ , consider the problem of maximizing

$$J_k^0(\eta, c(\cdot)) := \int_0^{+\infty} e^{-\rho t} U_1(c(t)) dt,$$

over  $\mathcal{C}_{ad}^k(\eta)$  and denote by  $V_k^0$  the associated value function.

Finally consider the problem of maximizing over the set  $\mathcal{C}_{ad}^0(\eta)$  the functional

$$J_0^n(\eta, c(\cdot)) := \int_0^{+\infty} e^{-\rho t} (U_1(c(t)) + U_2^n(y(t; \eta, c(\cdot)))) dt,$$

and denote by  $V_0^n$  the associated value function.

For fixed  $n \in \mathbb{N}$ , the problems  $V_k^n$  approximate, when  $k \rightarrow \infty$ , the problem  $V_0^n$  in the sense of Proposition 4.6, i.e. we are able to produce  $k_{\varepsilon, n}, M_{\varepsilon, n}$  large enough to make the strategy  $c_{k_{\varepsilon, n}, M_{\varepsilon, n}}(\cdot)$  defined as in (43) admissible and  $\varepsilon$ -optimal for the problem  $V_0^n(\eta)$ .

**Proposition 4.8.** *Let  $\eta \in H_{++}$ , let  $k_{\varepsilon, n}, M_{\varepsilon, n}, c_{k_{\varepsilon, n}, M_{\varepsilon, n}}(\cdot)$  as above. For every  $\varepsilon > 0$  we can find  $n_\varepsilon$  such that*

$$\lim_{\varepsilon \downarrow 0} V_{k_{\varepsilon, n_\varepsilon}, n_\varepsilon}^{n_\varepsilon}(\eta) = V_0^0(\eta). \quad (53)$$

Moreover the controls  $c_{k_{\varepsilon, n_\varepsilon}, M_{\varepsilon, n_\varepsilon}}(\cdot)$  defined as in (43) are admissible and  $3\varepsilon$ -optimal for the problem  $V_0^0(\eta)$ .

**Proof.** Let  $\varepsilon > 0$  and consider the strategies  $c^\varepsilon(\cdot)$  and  $c_M^\varepsilon(\cdot)$  defined as in the part (i) of the proof of Proposition 4.6. Notice that actually  $M = M(\varepsilon, n) =: M_n^\varepsilon$ . Notice also that by definition of  $k_{\varepsilon, n}, M_n^\varepsilon$  we have  $x_{k_{\varepsilon, n}}(\cdot; \eta, c_{M_n^\varepsilon}^\varepsilon(\cdot)) \geq \nu_1$  and that (37) in particular implies

$$\frac{1}{\rho} e^{-\rho M_n^\varepsilon} (\bar{U}_1 - U_1(0)) \leq \varepsilon.$$

Take  $n_\varepsilon \in \mathbb{N}$  such that  $1/n_\varepsilon < \nu_1$  (notice that  $\nu_1$  depends on  $\varepsilon$  and does not depend on  $n$ ). Then, since  $U_2^{n_\varepsilon} \equiv 0$  on  $[\nu_1, +\infty)$ , we can write

$$\begin{aligned} V_0^0(\eta) - \varepsilon &\leq J_0^0(\eta; c^\varepsilon(\cdot)) = \int_0^{+\infty} e^{-\rho t} U_1(c^\varepsilon(t)) dt \\ &= \int_0^{+\infty} e^{-\rho t} \left[ U_1(c^\varepsilon(t)) + U_2^{n_\varepsilon}(x_{k_{\varepsilon, n_\varepsilon}}(t; \eta, c_{M_{n_\varepsilon}^\varepsilon}^\varepsilon(\cdot))) \right] dt \\ &\leq J_{k_{\varepsilon, n_\varepsilon}, n_\varepsilon}^{n_\varepsilon}(\eta, c_{M_{n_\varepsilon}^\varepsilon}^\varepsilon(\cdot)) + \frac{1}{\rho} e^{-\rho M_{n_\varepsilon}^\varepsilon} (\bar{U}_1 - U_1(0)) \leq V_{k_{\varepsilon, n_\varepsilon}, n_\varepsilon}^{n_\varepsilon}(\eta) + \varepsilon, \end{aligned} \quad (54)$$

so that

$$\liminf_{\varepsilon \downarrow 0} V_{k_{\varepsilon, n_\varepsilon}, n_\varepsilon}^{n_\varepsilon}(\eta) \geq V_0^0(\eta). \quad (55)$$

On the other hand the strategies  $c_{k_{\varepsilon, n}, M_{\varepsilon, n}}(\cdot)$  defined in (43) are admissible for the problem  $V_0^0$  (since the state equation related to  $V_0^n$  and to  $V_0^0$  is the same) and  $c_{k_{\varepsilon, n}, M_{\varepsilon, n}}(\cdot)$  is  $\varepsilon$ -optimal for  $V_{k_{\varepsilon, n}}^n(\eta)$  for every  $n \in \mathbb{N}$ . Therefore

$$V_{k_{\varepsilon, n_\varepsilon}, n_\varepsilon}^{n_\varepsilon}(\eta) - \varepsilon \leq J_{k_{\varepsilon, n_\varepsilon}, n_\varepsilon}^{n_\varepsilon}(\eta; c_{k_{\varepsilon, n_\varepsilon}, M_{\varepsilon, n_\varepsilon}}(\cdot)) \leq J_{k_{\varepsilon, n_\varepsilon}, n_\varepsilon}^0(\eta; c_{k_{\varepsilon, n_\varepsilon}, M_{\varepsilon, n_\varepsilon}}(\cdot)) = J_0^0(\eta; c_{k_{\varepsilon, n_\varepsilon}, M_{\varepsilon, n_\varepsilon}}(\cdot)) \leq V_0^0(\eta), \quad (56)$$

which shows

$$\limsup_{\varepsilon \downarrow 0} V_{k_{\varepsilon, n_\varepsilon}, n_\varepsilon}^{n_\varepsilon}(\eta) \leq V_0^0(\eta). \quad (57)$$

Combining (55) and (57) we get (53). Combining (54) and (56) we get

$$V_0^0(\eta) \leq J_0^0(\eta; c_{k_{\varepsilon, n_\varepsilon}, M_{\varepsilon, n_\varepsilon}}(\cdot)) + 3\varepsilon,$$

i.e. the last claim.  $\square$

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